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Wronski Brackets and the Ferris Wheel

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We connect the Bayesian order on classical states to a certain Lie algebra on $C^\infty[0, 1]$. This special Lie algebra structure, made precise by an idea we introduce called a *Wronski bracket*, suggests new phenomena the Bayesian order naturally models. We then study Wronski brackets on associative algebras, and in the commutative case, discover the beautiful result that they are equivalent to derivations.

KEY WORDS: quantum information; Lie algebras; domain theory.

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1. INTRODUCTION

In Coecke and Martin (2002), a partial order on classical states Δ^n was introduced called the Bayesian order. The Bayesian order has the seemingly rare property that it extends to quantum states Ω^n in such a way that several traits desirable to computer scientists and physicists alike are preserved. One of these is that the classical and quantum logics of Birkhoff and von Neumann arise as the set of irreducible elements of Δ^n and Ω^n, respectively.

In this paper, we characterize the Bayesian order in a manner reminiscent of causal relations in relativity: by saying that two states compare iff they are joined by a certain type of curve. The type of the curve is expressed using a new idea called a Wronski bracket. Wronski brackets are special types of Lie brackets that do not appear to have been studied before. The definitive property of a Wronski bracket on a commutative algebra $\mathcal{A}$ is that

$$[ax, by] = (ab)[x, y] + [a, b](xy)$$

for all $a, b, x, y \in \mathcal{A}$, which turns out to be a stronger form of the Jacobi identity. From an algebraic viewpoint this is interesting because we carry two distinct multiplications, and our axiom serves to relate them. Wronski brackets have many important uses in physics and mathematics, and one of the major results of this

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paper is that Wronski brackets on commutative algebras with identity are in 1–1 correspondence with derivations (i.e., on smooth manifolds derivations are equivalent to vector fields).

Wronski brackets arise in so many different contexts that by combining them in a meaningful way we discover new phenomena naturally modeled by the Bayesian order. One of these, the postdoc ferris wheel, teaches us what the quintessential monotone process in the Bayesian order is like. Due to space limitations, some proofs are given in the report (Martin, 2004). Lemma 4 and all results after that are new.

2. DIFFERENTIABLE CURVES

For an integer $n \geq 2$, the *classical states* are

$$\Delta^n := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1 \right\}.$$

A classical state $x \in \Delta^n$ is *pure* when $x_i = 1$ for some $i \in \{1, \ldots, n\}$; we denote such a state by e_i. States that are not pure are called *mixed*. If we know x and by some means determine outcome i is not possible, our knowledge improves to

$$p_i(x) = \frac{1}{1 - x_i}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \in \Delta^n,$$

where $p_i(x)$ is obtained by first removing x_i from x and then renormalizing. The partial mappings which result, $p_i : \Delta^{n+1} \rightharpoonup \Delta^n$ with $\mathrm{dom}(p_i) = \Delta^{n+1} \setminus \{e_i\}$, are called the *Bayesian projections* and lead one to the following relation on classical states.

Definition 2.1. For $x, y \in \Delta^{n+1}$,

$$x \sqsubseteq y \equiv (\forall i)(x, y \in \mathrm{dom}(p_i) \Rightarrow p_i(x) \sqsubseteq p_i(y)).$$

For $x, y \in \Delta^2$,

$$x \sqsubseteq y \equiv (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1).$$

The relation $\sqsubseteq$ on Δ^n is called the *Bayesian order*.
The Bayesian order was introduced in Coecke and Martin (2002) where the following is proven.

Theorem 2.2. *$(\Delta^n, \sqsubseteq)$ is a domain with least element $\bot := (1/n, \ldots, 1/n)$ and* $\max(\Delta^n) = \{e_i : 1 \leq i \leq n\}$.

A *domain* is a partially ordered set with intrinsic notions of completeness and approximation. The exact definition is given in Coecke and Martin (2004).

The equality $\max(\Delta^n) = \{e_i : 1 \leq i \leq n\}$ follows from two crucial aspects of the Bayesian order: (i) $x \sqsubseteq y \Rightarrow (\exists i)\, x_i = x^+ \;\&\; y_i = y^+$ and (ii) $x \sqsubseteq e_i \Leftrightarrow x_i = x^+$, where we set $x^+ = \max\{x_i : 1 \leq i \leq n\}$ for $x \in \Delta^n$.

The Bayesian order has a more direct description: The *symmetric formulation* of Coecke and Martin (2002). Let $S(n)$ denote the group of permutations on $\{1, \ldots, n\}$ and

$$\Lambda^n := \{x \in \Delta^n : (\forall i < n)\, x_i \geq x_{i+1}\}$$

denote the collection of *monotone* decreasing classical states.

Theorem 2.3. *For $x, y \in \Delta^n$, we have $x \sqsubseteq y$ iff there is a permutation $\sigma \in S(n)$ such that $x \cdot \sigma, y \cdot \sigma \in \Lambda^n$ and*

$$(x \cdot \sigma)_i (y \cdot \sigma)_{i+1} \leq (x \cdot \sigma)_{i+1} (y \cdot \sigma)_i$$

for all i with $1 \leq i < n$.

Thus, $(\Delta^n, \sqsubseteq)$ can be thought of as $n!$ many copies of the domain $(\Lambda^n, \sqsubseteq)$ identified along their common boundaries, where $(\Lambda^n, \sqsubseteq)$ is $x \sqsubseteq y \equiv (\forall i < n)\, x_i y_{i+1} \leq x_{i+1} y_i$. It should be remarked though that the problems of ordering Λ^n and Δ^n are very different, with the latter being far more challenging if one also wants to consider quantum mixed states.

Our first observation is that movement in the Bayesian order implies differentiability (a.e.). A curve $\pi : [0, 1] \to \Lambda^n$ can be written as $\pi = (\pi_1, \ldots, \pi_n)$ where the functions $\pi_i : [0, 1] \to [0, 1]$ satisfy $\pi_i \geq \pi_{i+1}$ for $i < n$ and

$$\sum_{i=1}^{n} \pi_i(t) = 1$$

for all $t \in [0, 1]$.

Lemma 2.4. *If $x \sqsubseteq y$ in Λ^n, then*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$

for all $1 \leq k \leq n$. The converse is false.

Proof: Because $x \sqsubseteq y$ in Λ^n, there is an integer $1 < m \leq n$ such that $x_i \leq y_i$ for $i < m$ and $x_i \geq y_i$ for $i \geq m$. Then for $k < m$ the claim is obvious, while for $k \geq m$ we have

$$\sum_{i=1}^{k} x_i = 1 - \sum_{i=k+1}^{n} x_i \leq 1 - \sum_{i=k+1}^{n} y_i = \sum_{i=1}^{k} y_i,$$

using $x_i \geq y_i$ for $i \geq m$. $\square$

The property of the Bayesian order in Lemma 2.4 defines an order $\leq$ on Λ^n in its own right,

$$x \leq y \equiv (\forall k \in \{1, \ldots, n\}) \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$$

called *Majorization* (Alberti and Uhlmann, 1982; Marshall and Olkin, 1979; Muirhead, 1903). It too yields a domain $(\Lambda^n, \leq)$ as noted in (Martin, 2003); however, it has no natural extension to all of Δ^n. Despite this sharp difference, an increasing curve in either order has a decent amount of analytic structure.

Proposition 2.5. *If $\pi = (\pi_1, \ldots, \pi_n) : [0, 1] \to (\Lambda^n, \leq)$ is an increasing curve, then each π_i is of bounded variation. Thus,*

(i) *the curve π has a 'length';*
(ii) *it is continuous except on a countable set;*
(iii) *it is differentiable except on a set of measure zero.*

Proof: Define $f_k : [0, 1] \to [0, 1]$ by

$$f_k(t) = \sum_{i=1}^{k} \pi_i(t)$$

for $k \geq 0$. By the definition of $\leq$, the f_k are monotone increasing functions, where we note that $f_k = 0$ for $k = 0$. Since $\pi_i = f_i - f_{i-1}$, the map π_i is the difference of monotone increasing maps and thus of bounded variation. This establishes (i), (ii), and (iii). $\square$

From the qualitative *follows* differentiability. We study differentiable curves, which as we shall see, will take us naturally to the Wronski bracket.

3. THE WRONSKI BRACKET

Let X denote the set of differentiable real valued functions defined on [0, 1]. For $x, y \in X$,

$$[x, y] := \dot{x}y - \dot{y}x.$$

This is called the *Wronski bracket*.

Lemma 3.1. *Let $x, y, z \in X$. Then*

(i) $y[x, z] = z[x, y] + x[y, z]$,
(ii) $\dot{y}[x, z] = \dot{z}[x, y] + \dot{x}[y, z]$,
(iii) $y[x \dot{,} z] = z[x \dot{,} y] + x[y \dot{,} z]$.

In addition, $[x, 1] = \dot{x}$.

Proof: (iii) Use the identity $[x \dot{,} y] = \ddot{x}y - \ddot{y}x$. $\square$

The perfection of form noted in the last lemma suggests that X with $[\,,\,]$ has more structure than the usual Lie algebra. Nevertheless:

Lemma 3.2. $[\,,\,]$ *is a Lie bracket.*

(i) $[\,,\,]$ *is bilinear,*
(ii) $[x, y] = -[y, x]$,
(iii) *The Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

holds.

Thus, the smooth maps in X with $[\cdot, \cdot]$ form a Lie algebra.

Proof: (iii) What is beautiful here is that the Jacobi identity follows from (ii) and (iii) of Lemma 3.1. $\square$

The next result shows that there is a relationship between the standard order on curves and the Wronski bracket.

Proposition 3.3. *Let $f, g, h : [0, 1] \to [0, \infty)$ be differentiable maps with $f \geq g \geq h$.*

(i) *If $[f, g] \geq 0$ and $[g, h] \geq 0$, then $[f, h] \geq 0$.*
(ii) *If $[f, g] = 0$ and $[g, h] = 0$, then $[f, h] = 0$.*

Proof: (i) Let $s \in [0, 1]$. If $g(s) > 0$, then the equation

$$g[f, h] = h[f, g] + f[g, h]$$

of Lemma 3.1(i) applies, which immediately yields $[f, h](s) \geq 0$. Thus, we need only verify the assertion for $g(s) = 0$.

But if $g(s) = 0$, then we must also have $h(s) = 0$. For $s \in (0, 1)$, we have $\dot{h}(s) = 0$, which immediately gives $[f, h](s) = 0$. Likewise, $f(s) = 0$ also gives

$[f, h](s) = 0$, so we may assume that (i) $s \in \{0, 1\}$, (ii) $f(s) > 0$, and (iii) $g(s) = h(s) = 0$. If $s = 1$, then

$$\dot{h}(s) = \lim_{t \to 1^-} \frac{h(t)}{t - 1} \le 0,$$

so $[f, h](s) = -\dot{h}(s) f(s) \ge 0$. If $s = 0$, then $\dot{h}(s) \ge 0$. Since $g(s) = 0$, $[f, g](s) = -\dot{g}(s) f(s) \ge 0$, so $f(s) > 0$ yields $\dot{g}(s) \le 0$. But $g \ge h$ gives

$$\dot{h}(s) = \lim_{t \to 0^+} \frac{h(t)}{t} \le \lim_{t \to 0^+} \frac{g(t)}{t} = \dot{g}(s) \le 0,$$

and hence $\dot{h}(s) = 0$. Then $[f, h](s) = 0$, which finishes the proof.

(ii) Applying (i) to $f \ge g \ge h$ shows $[f, h] \ge 0$. But we can also apply (i) to $f \ge g \ge g - h$ since $[g, g - h] = 0$, which gives

$$[f, g - h] \ge 0$$

and that is exactly $[f, h] \le 0$. $\square$

The *Wronski bracket* $[x, y] = \dot{x} y - \dot{y} x$ determines when two solutions of a second order equation are independent, the area swept out by a curve in the plane, angular momentum in mechanics, its sign classifies the direction of motion as being either clockwise or counterclockwise, and it arises in model theory as the only known example of a 'stable' infinite dimensional Lie algebra. It is also intimately connected to the Bayesian order.

Theorem 3.3. *A differentiable curve $\pi = (\pi_1, \ldots, \pi_n) : [0, 1] \to (\Lambda^n, \sqsubseteq)$ is increasing iff $[\pi_i, \pi_{i+1}] \ge 0$ for all $i < n$.*

Proof: (i) $\Rightarrow$ (ii): To prove $[\pi_i, \pi_{i+1}](s) \ge 0$, we can assume $\pi_i(s) > 0$ (otherwise, we trivially have $[\pi_i, \pi_{i+1}](s) = 0$). By the continuity of π_i, there must be an open interval $(a, b) \subseteq \mathbb{R}$ containing s such that $\pi_i > 0$ on $U := (a, b) \cap [0, 1]$. By the monotonicity of π, the map π_{i+1}/π_i is monotone decreasing on U. Since it is also differentiable, its derivative cannot be positive. But

$$\dot{\left(\frac{\pi_{i+1}}{\pi_i}\right)} = \frac{[\pi_{i+1}, \pi_i]}{\pi_i^2} \le 0,$$

which makes it clear that $[\pi_i, \pi_{i+1}] = -[\pi_{i+1}, \pi_i] \ge 0$.

(ii) $\Rightarrow$ (i): We need to show that

$$\pi_i(s)\pi_{i+1}(t) \le \pi_{i+1}(s)\pi_i(t)$$

whenever $s \le t$ and $1 \le i < n$. We can assume $s < t$. In the simple case, $\pi_i > 0$ on $[s, t]$, and then as before

$$\dot{\left(\frac{\pi_{i+1}}{\pi_i}\right)} = \frac{[\pi_{i+1}, \pi_i]}{\pi_i^2} = -\frac{[\pi_i, \pi_{i+1}]}{\pi_i^2} \le 0$$

which means that π_{i+1}/π_i is monotone decreasing on $[s, t]$. In particular, $(\pi_{i+1}/\pi_i)(s) \geq (\pi_{i+1}/\pi_i)(t)$, which finishes this case.

Suppose now that $\pi_i(c) = 0$ for some $c \in [s, t]$. We claim that $\pi_i(t) = 0$. For the proof, a simple induction based on Proposition 3.3 gives $[\pi_i, \pi_j] \geq 0$ for $1 \leq i \leq j \leq n$. This means that $f_k : [0, 1] \to [0, 1]$ given by

$$f_k = \sum_{i=1}^{k} \pi_i$$

is monotone increasing since

$$\dot{f}_k = [f_k, 1] = \left[f_k, f_k + \sum_{j>k} \pi_j \right] = 0 + \sum_{j>k} [f_k, \pi_j] = \sum_{j>k} \sum_{i=1}^{k} [\pi_i, \pi_j] \geq 0.$$

Because $\pi(c)$ is a monotone state, $\pi_i(c) = 0$ implies that $f_i(c) = 1$. But if $\pi_i(t) > 0$, then the monotonicity of f_i gives $1 = f_i(c) < f_i(t)$, which contradicts the fact that $\pi(t)$ is a classical state. Then $\pi_i(t) = 0$, so $\pi_i \geq \pi_{i+1}$ gives $\pi_{i+1}(t) = 0$, which proves the desired inequality holds. □

The previously mentioned uses of the Wronski bracket provide many new ways to explain the Bayesian order. We can also characterize it in analytic terms.

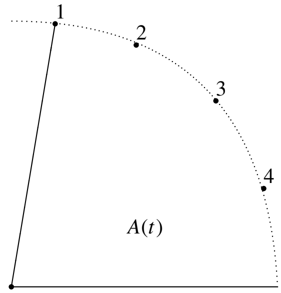
Corollary 3.5. *For $x, y \in \Lambda^n$, $x \sqsubseteq y$ iff there is a differentiable curve $\pi : [0, 1] \to \Lambda^n$ from $x = \pi(0)$ to $y = \pi(1)$ with $[\pi_i, \pi_{i+1}] \geq 0$ for all $i < n$.*

Proof: Given $x \sqsubseteq y$, let $\pi : [0, 1] \to \Lambda^n$ be the straight line path from x to y. The other direction follows from the last theorem. □

An interesting point is that for a differentiable curve $\pi : [0, 1] \to \Lambda^n$, we have $[\pi_i, \pi_j]$ constant for all $1 \leq i, j \leq n$ iff $\pi(t) = (1 - t) \cdot \pi(0) + t \cdot \pi(1)$. Finally, all results in this section apply equally well to curves $\pi : [a, b] \to \Lambda^n$. We simply took $a = 0$ and $b = 1$ because we like them.

4. THE POSTDOC FERRIS WHEEL

The connection between the Bayesian order and the Wronski bracket suggests the quintessential example of a monotone process. Imagine four postdocs on a ferris wheel at the start of *a new revolution*:



The question we want to ask is

What is the probability that postdoc i wins?

where by 'wins' we mean that postdoc i completes the revolution before all others. This seems like a strange and uninteresting question until we learn more about the postdocs:

Postdoc 1: Believes there's only one place to go from the top.
Postdoc 2: Just wants to enjoy the ride while it lasts.
Postdoc 3: Thinks that every *revolution* needs at least one martyr.
Postdoc 4: His funding was just renewed for *two* more months—wow!

So for various reasons, it is possible that some of the postdocs may jump from the ferris wheel before the current revolution is completed. How then can we calculate the probability that postdoc i wins?

First, let us calculate the probability that one of them jumps. The only information we have is that based on observation (literally "watching" the wheel move), so the probability that i jumps is determined by the percentage of total area swept out by the line joining the origin to i:

$$P(i \text{ jumps})(t) = 1 - a_i(t) := 1 - \frac{A(t-(i-1)\varepsilon)}{A}$$

where A is the total area swept out by one revolution of the wheel, $A(t)$ is the area swept out by postdoc 1 after t units,

$$A(t) = \frac{1}{2}\int_0^t [y, x]\, ds,$$

and the coordinates of postdoc 1 are $(x(t), y(t))$. Notice that we take the coordinates of postdoc i to be $(x(t-(i-1)\varepsilon), y(t-(i-1)\varepsilon))$ where $0 < \varepsilon < 1$ is a constant. Using that the probability of Y given X is

$$P(Y \mid X) = \frac{P(Y \& X)}{P(X)},$$

we get

$$\begin{aligned}
P(i \text{ wins}) &= P(i \text{ does not jump \& all } j \text{ jump for } j < i) \\
&= P(i \text{ does not jump} \mid \text{all } j < i \text{ jump}) \cdot P(\text{all } j < i \text{ jump}) \\
&= a_i \cdot P(\text{all } j < i \text{ jump}) \\
&= a_i \cdot \prod_{j=1}^{i-1} (1 - a_j)
\end{aligned}$$

assuming for the last equality that the postdocs jump independent of one another. These probabilities, when normalized by $P := \Sigma P(i \text{ wins})$, define a curve π into Λ^4 given by $\pi_i = P(i \text{ wins})/P$. After enough time elapses, π is *monotone*!

To see this, suppose that the coordinates of postdoc 1 have been parameterized over [0, 1] as

$$x(t) = r\cos(2\pi t) \; : \; y(t) = r\sin(2\pi t)$$

where r is the radius of the ferris wheel. Then π is now defined on $[3\varepsilon, 1]$.

Lemma 4.1. *The curve π is monotone increasing on* $[2\varepsilon + \sqrt{\varepsilon}, 1]$.

Proof: Using the identity $[ax, ay] = a^2[x, y]$, bilinearity and Theorem 3, π will be monotone if $[a_i, a_{i+1}] + a_i^2 \dot{a}_{i+1} \geq 0$ for all i. Since each a_i is a translation of a_1, we first consider the case $i = 1$. We get

$$a_1(t) = t, \qquad a_2(t) = t - \varepsilon, \qquad A = \pi r^2$$

so $[a_1, a_2] + a_1^2 \dot{a}_2 \geq 0$ iff $t \geq \sqrt{\varepsilon}$. Applying this result to a_i and a_{i+1}, we get $[a_i, a_{i+1}] + a_i^2 \dot{a}_{i+1} \geq 0$ iff $t \geq (i-1)\varepsilon + \sqrt{\varepsilon}$. Setting $i = 3$ finishes the proof. □

Now suppose the wheel is turning when at some point $t \in [2\varepsilon + \sqrt{\varepsilon}, 1]$, postdoc i jumps. At this instant, our knowledge of who will win becomes

$$p_i \circ \pi : [t, 1] \to \Lambda^3$$

where

$$p_i(x) = \frac{1}{1 - x_i}(x_1, \ldots, \hat{x}_i, \ldots, x_4) \in \Delta^3$$

is a Bayesian projection that first removes x_i from x and then renormalizes. This new curve $p_i \circ \pi$ *is also monotone increasing* because π is monotone increasing in the *Bayesian order*. Why does not π increase over the entire interval $[3\varepsilon, 1]$?

We do not experience an increase in information until $t = 2\varepsilon + \sqrt{\varepsilon}$, which means that if you were watching the ferris wheel, then from the moment that postdoc 4 crossed the x-axis ($t = 3\varepsilon$), you would have to wait

$$(2\varepsilon + \sqrt{\varepsilon}) - 3\varepsilon = \sqrt{\varepsilon} - \varepsilon$$

units of time until you experienced an increase of information (according to the Bayesian order). This it seems should be regarded as "the amount of time required for information to be converted into credible belief," on the grounds that it takes time for credible belief to be established.

Another example in the same spirit as the ferris wheel would be the state of a queue of processes waiting to exploit some resource. At any moment, a process may get tired of waiting and elect to remove itself from the queue. For instance, if a number of users are waiting to download a large file. What is important, though, is to specify that a user has no knowledge of their position in the queue—otherwise, taking the probabilities for jumping to be independent may not make much sense.

The Wronski bracket arises in many different contexts and combining these (conservation of angular momentum, the Bayesian order) has led to new phenomena modeled by the Bayesian order, such as a ferris wheel of maladjusted postdocs, or a queue of impatient processes. What we want to do now is identify some property it has that can help explain why it is so 'special'. In proving Lemma 4.1, we encounter a formula which provides a hint:

$$[ax, ay] = a^2[x, y].$$

Pragmatically, this tells us that renormalization does not affect monotonicity in the Bayesian order. This is a special case of a more general property

$$[ax, by] = (ab)[x, y] + [a, b](xy)$$

which serves to *characterize* the Wronski bracket.

5. WRONSKI BRACKETS AND DERIVATIONS

An *algebra* over the field of real numbers is a real vector space $(A, +)$ with a bilinear multiplication $\cdot : A^2 \to A$. An algebra is associative if $\cdot$ is associative, commutative if $\cdot$ is commutative and has an identity if $\cdot$ has an identity. Let A be an associative algebra with identity. Its commutator is

$$\langle x, y\rangle := xy - yx$$

for $x, y \in A$. Here is one way to define Wronski brackets on arbitrary associative algebras with identity.

Definition 5.1. A *Wronski bracket* is an antisymmetric bilinear map $[\ ,\ ] : A^2 \to A$ such that

$$[ax, by] = (ab)[x, y] + [a, b](xy) + a\langle b, x\rangle[y, 1] + [1, a]\langle b, x\rangle y$$

for all $a, b, x, y \in A$.

An equivalent definition is to replace antisymmetry above by the axiom $[x, 1] + [1, x] = 0$.

Definition 5.2. A *derivation* is a linear map $d : A \to A$ such that

$$d(xy) = dx \cdot y + x \cdot dy$$

for all $x, y \in A$.

Proposition 5.3. *For each derivation d with $\langle dx, y\rangle = \langle x, dy\rangle$,*

$$[x, y] := dx \cdot y - x \cdot dy$$

is a Wronski Bracket.

Proposition 5.4. *For each Wronski bracket* $[\,,\,]$

$$dx := [x, 1]$$

is a derivation with $\langle dx, y\rangle = \langle x, dy\rangle$ and

$$[x, y] = dx \cdot y - x \cdot dy$$

for all $x, y \in A$.

So there is a map from derivations to Wronski brackets $d \mapsto [\,,\,]_d$ and another from Wronski brackets to derivations $[\,,\,] \mapsto d_{[\,,\,]}$.

Theorem 5.5. *There is a one-to-one correspondence between Wronski brackets and derivations $d : A \to A$ that satisfy $\langle dx, y\rangle = \langle x, dy\rangle$.*

For the case we are most interested in, let A be a commutative algebra with identity. Because the commutator is now identically zero, the axiom for the Wronski bracket simplifies to

$$[ax, by] = (ab)[x, y] + [a, b](xy)$$

and $\langle dx, y\rangle = \langle x, dy\rangle$ holds for all derivations $d : A \to A$. The axiom for Wronski brackets can be used to derive the equations shown valid in the case $A = C^\infty[0, 1]$, including the Jacobi identity.

Definition 5.6. A *Lie bracket* is a bilinear, antisymmetric $[\,,\,] : A^2 \to A$ with

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in A$.

Given a Wronski bracket [,] on A, we write

$$\dot{x} := [x, 1]$$

for its associated derivation. Then

$$[x, y] = \dot{x}y - x\dot{y}$$

for all $x, y \in A$.

Proposition 5.7. *A Wronski bracket* [,] *satisfies*

(i) $y[x, z] = z[x, y] + x[y, z]$,
(ii) $\dot{y}[x, z] = \dot{z}[x, y] + \dot{x}[y, z]$,
(iii) $y\dot{[x, z]} = z\dot{[x, y]} + x\dot{[y, z]}$.

Thus, a Wronski bracket is a Lie bracket.

An important point: It was never clear before why the Lie structure of the Wronski bracket mattered. From the present point of view, the Jacobi identity is a generalization of the Leibniz identity (the product rule for derivatives). To the best of our knowledge, this connection between the two has not been made.

Example 5.8. (*Vector fields*) If M is a smooth manifold, a vector field is a derivation on the algebra of smooth real-valued functions $C^{\infty}(M)$. Thus, Wronski brackets on $C^{\infty}(M)$ are precisely vector fields.

A natural question is whether there is a connection between Wronski brackets and the crucial *Poisson brackets*: Lie brackets [,] with the property that each $[x, \cdot]$ defines a derivation, for $x \in A$. The Wronski bracket $[x, y] = \dot{x}y - x\dot{y}$ is not a Poisson bracket. Whether there is some other way of relating the two ideas, we do not know (but would be very interested in hearing from someone who does know).

6. AN ALGEBRAIC FORMULATION OF MONOTONE PROCESS

It may seem that we have wandered off course pretty far from the domain of classical states, so we should point out before ending that the extra structure in an algebra combined with what we have learned about Wronski brackets allows one to adopt an abstract view of the Bayesian order that may be applicable in other settings. First, if A is an algebra, we can define the *spectrum* of $x \in A$ to be

$$\sigma x := \{\alpha \in \mathbb{R} : x - \alpha \cdot 1 \text{ has no multiplicative inverse}\}$$

and then say $x \geq 0$ iff $\sigma x \subseteq [0, \infty)$. Using this, we can define a relation

$$x \leq y \equiv (y - x) \geq 0.$$

In the case $A = C^\infty(M)$, we will get $\sigma x = x(M)$, and $\leq$ is the usual pointwise order on functions. Given a commutative algebra A and a derivation $d : A \to A$, we take $\leq$ and call a vector $v \in A^n$ with $v_i \geq v_{i+1}$ *monotone* if

$$(\forall i)\,[v_i, v_{i+1}] \geq 0$$

where [,] is the canonical Wronski bracket associated to d. We can also require $\Sigma v_i = 1$ in general, though in some cases this may be difficult to justify (perhaps something like $\Sigma dv_i = 0$ is better). In the case $A = C^\infty[0, 1]$, a monotone $v \in V^n$ with $\Sigma v_i = 1$ is then an increasing curve in the Bayesian order. It is possible to distinguish states from curves in this way as well, by calling $v \in A^n$ a *state* if $dv_i = 0$. Properties like those in Proposition 3.3 are probably the kind one needs to develop these notions more fully.

One thing ideas along this line seem to offer is a completely different way of thinking about (the Bayesian) order, an *implicit* description of it. Instead of $x \sqsubseteq y$, we characterize events that begin with x and end at y, i.e., *processes* that cause a change of state from x to y. In some cases, it may not be necessary to know x and y, but only processes which connect them.

7. CLOSING REMARKS

Is every Wronski bracket a Lie bracket, in general? This might be a good way to test a noncommutative definition of Wronski bracket. It may be possible to obtain more pleasing results in the noncommutative case, but in the case we are most interested in here (the commutative one), we do not believe that it is. It would be good to consider an algebra of operators on a Hilbert space.

A more realistic model of the ferris wheel might assume the postdocs are well adjusted and instead that there is a person nearby with a control panel having buttons labeled 1, 2, 3, 4. The person, called "professor," has the option of pressing button i, and if he does, this results in postdoc i being immediately thrown from the ferris wheel. Unfortunately, the present author was born without the ability to take this model seriously, so we have assumed that the postdocs are governed by free will.

We do not have to study maladjusted postdocs on ferris wheels of course. It could be a ferris wheel of unhappy professors concerned that one of their favorite postdocs is going to commit academic suicide before realizing his true potential. But, regardless of the choice, the crucial point should always be not to bore the reader (or the writer) to the desperate point of sleep. Those who have trouble keeping us awake will not like this.

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